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On the identification problems in products of cycles

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Abstract

A set of subgraphs $G_1,\dots,G_k$ in a graph G is said to identify the vertices v (resp. edges e) if the sets $\{j : v \in G_j\}$ (resp. $\{j : e \in G_j\}$) are all nonempty and different. In this paper we prove upper bounds for the smallest cardinalities of vertex and edge identifying collections of cycles and closed walks. In particular, we prove that the smallest cardinality of edge identifying collection of closed walks in the binary Hamming space is $n + \lfloor \log_2 n \rfloor$. We also consider the identification of paths of length two.

Keywords: Hamilton decompositions; Identifying subsets; Interconnection networks

1. Introduction

The general definition of an identifying collection of subsets is as follows.

Definition 1. A collection $A_1, A_2, \dots, A_k$ of subsets of a set S is called *identifying* if the sets $\{i : x \in A_i\}$, where $x \in S$, are nonempty and different.

If we allow that there is one (but at most one) element y such that $\{i : y \in A_i\} = \emptyset$, then the collection is said to be *separating*.

An obvious test whether or not $A_1, \dots, A_k$ is an identifying collection of subsets of an s-element set S, is provided by a $k \times s$ matrix $\mathbf{A}$ whose columns are indexed with elements of S and whose rows are the characteristic vectors of the subsets A_i. Namely,

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$A_1, \ldots, A_k$ is identifying if and only if the columns of $\mathbf{A}$ are nonzero and distinct. Even if the sets A_i are required to have some kind of structure, $\mathbf{A}$ can sometimes be used to solve existence questions.

Identifying collections of subsets (or codes) were introduced in [15] and since that paper a lot of research has been done in this area. A prominent part of the papers published on identification problems considers the binary Hamming space and the subsets A_i are in this case usually required to be Hamming balls of radius one. The interested reader should consult e.g. the papers [15,5,4].

Identification problems have been considered also in various other graphs and also different identifying subsets have been studied, see the articles [8,13,14].

The identification problem is related to so-called locating-dominating sets, see Ref. [9]; for a recent paper, see [19]. A strong connection to (multiple) coverings was established in [17]. It is also connected to fault-tolerant message routing problems studied e.g. in [20,21].

The study of identifying collections of subsets is motivated by fault diagnosis in multiprocessor architectures, i.e., these collections are used for locating a malfunctioning processor. As usual, a multiprocessor architecture is represented as a graph. Each vertex corresponds to a processor and each edge represents a link between two processors. In the simplest variant we assume that at most one of the processors is malfunctioning, and we wish to identify it (or to find that none of them is malfunctioning). We use the following scheme. Let S be the set of vertices. We choose the identifying subsets $A_1, \ldots, A_k$ of S. The set of processors in each A_i is checked and we get YES/NO answers telling whether or not any problems were detected in A_i. Based on these k YES/NO answers we are able to identify the malfunctioning processor (or to tell that there is none). It is natural to pose various constraints on the sets A_i, e.g., to require that they are balls with respect to the graphic distance, or that they are cycles or closed walks. If the sets A_i form cycles (or closed walks), the checking can be done by sending simultaneous signals from some nodes to the system. The malfunctioning processor is determined by the signals which come back to the node from which it was sent. For more details, see [15,12,11,7].

In this paper we consider various identification problems in products of cycles. The identifying subsets are required to consist of vertices (or edges) of closed walks (or equivalently connected subgraphs). Earlier results on this problem can be found in [12,11,18].

For arbitrary sets we have the following obvious lower bound for the number of identifying subsets.

Theorem 2. *An identifying collection of subsets of an s-element set contains at least $\lceil \log_2(s+1) \rceil$ subsets.*

The lower bound given in Theorem 2 can always be attained if there is no requirement for the structure of the identifying subsets.

Lemma 3. *An s-element set S has an identifying collection $A_1, \ldots, A_k$ of subsets such that $k = \lceil \log_2(s+1) \rceil$ and $|A_i| = \lceil s/2 \rceil$ for all $i = 1, \ldots, k$.*

Proof. Clearly, there is an identifying collection of $\lceil \log_2(s+1) \rceil$ subsets. Assume that A_1, for instance, contains fewer than $\lceil s/2 \rceil$ elements. Then the corresponding matrix $\mathbf{A}$ has fewer columns beginning with 1 than columns beginning with 0. Thus, there is a column beginning with 0 whose $k-1$ last bits do not appear as $k-1$ last bits of any other column. Now the first bit of this column can be changed to 1. The same argument holds for zeros too (except that we have to avoid the zero column). $\square$

In Section 2 we study vertex identifying collections of subgraphs in products of graphs. We also present a new construction technique which uses identifying collections of subsets of smaller graphs to obtain identifying collections of subsets of bigger graphs.

In Section 3 we consider the edge identification problem. In particular, we show that the trivial lower bound given in Theorem 2 is attained in the binary Hamming space. We also consider the identification of turns, i.e., paths of length two.

2. Vertex identifying collections

For basic notions in graph theory we refer to [6]. Especially, we will use the following notions.

A *walk* in a graph is a finite nonnull sequence $W = v_0 v_1 \cdots v_n$ of vertices such that v_i and v_{i+1} are adjacent for all $i = 0, \ldots, n-1$. We also say that W is a walk *from v_0 to v_n*. The number n is the *length of W*. If $v = v_i$ for some i, we say that W *visits* the vertex v and write (by a slight abuse of notation) $v \in W$. If $v_0 = v_n$ and the length of W is positive, we say that the walk W is *closed*.

If all the vertices in a walk $W = v_0 v_1 \cdots v_n$ are distinct, then W is called a *path*. If W is closed, $n \geqslant 3$ and $v_i \neq v_j$ whenever $i \neq j$ and $i, j = 0, \ldots, n-1$, then W is a *cycle*. A path of length one is essentially nothing but an edge.

In a graph $G=(V,E)$ two vertices $u, v \in V$ are said to be *connected* if there is a walk, and thus also a path, from u to v in G. If u and v are connected, then the *distance* $d(u,v)$ is defined to be the length of a shortest path from u to v in G. If the graph G is *connected*, i.e., all pairs $u, v \in V$ are connected, the distance defines a metric on V.

A path which visits every vertex of a graph G is called a *Hamilton path* of G. Similarly, a *Hamilton cycle* of G is a cycle which visits every vertex of G. A graph is said to be *Hamiltonian* if it contains a Hamilton cycle (as a subgraph).

To identify vertices with cycles or closed walks is a routing problem. See [20,21] for a related problem. Yet another routing problem is considered in [3].

The smallest cardinality of a vertex identifying collection of cycles (resp. closed walks) in a graph G is denoted by $v(G)$ (resp. $v^*(G)$). If G has no such collection, we define $v(G) = \infty$ and $v^*(G) = \infty$.

Recall that the product of two graphs is defined as follows.

Definition 4. The *product* of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph

$$G_1 \times G_2 = (V_1 \times V_2, E),$$

where E consists of those pairs $\{(v_1,u_1),(v_2,u_2)\}$ for which either $v_1=v_2$ and $\{u_1,u_2\} \in E_2$ or $u_1=u_2$ and $\{v_1,v_2\} \in E_1$. The product of several graphs is defined similarly.

Our main interest in this paper are the products of cycles. Products of cycles are examples of so-called Cayley graphs, which are nowadays very popular models of interconnection networks, see [10].

Theorem 5. *Suppose that the graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ have Hamilton paths. Then*

(i) $v(G_1 \times G_2) \leqslant v(G_1)+v(G_2)$ *and*
(ii) $v^*(G_1 \times G_2) \leqslant v^*(G_1)+v^*(G_2)$.

Proof. First we prove that if $C=u_1u_2\dots u_nu_1$ is a cycle in G_1, then in the graph $G_1 \times G_2$ there is a cycle which consists of the vertices

$$C \times G_2 = \{(u,x) : u \in C, x \in V_2\}.$$

Let $P=v_1v_2\dots v_t$ be a Hamilton path in G_2. If t is even, then the vertices of the set $C \times G_2$ can be visited in the following order:

$$\begin{array}{ccccccccc}
(u_1,v_1) & \to & (u_2,v_1) & \to & (u_3,v_1) & \to & \cdots & \to & (u_n,v_1) \\
 & \to & (u_n,v_2) & \to & (u_{n-1},v_2) & \to & \cdots & \to & (u_2,v_2) \\
 & \to & (u_2,v_3) & \to & (u_3,v_3) & \to & \cdots & \to & (u_n,v_3) \\
 & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
 & \to & (u_n,v_t) & \to & (u_{n-1},v_t) & \to & \cdots & \to & (u_2,v_t) \\
 & \to & (u_1,v_t) & \to & (u_1,v_{t-1}) & \to & \cdots & \to & (u_1,v_1).
\end{array}$$

If t is odd, then only the next to last row changes: the indices of u increase from 2 to n.

By symmetry, we can form a cycle also from the vertices of the set

$$G_1 \times D = \{(y,v) : y \in V_1, v \in D\},$$

where D is any cycle in G_2.

Suppose that $C_1,\dots,C_s$ are vertex identifying cycles in G_1 and $D_1,\dots,D_t$ are vertex identifying cycles in G_2. Then the $s+t$ cycles with vertex sets $C_i \times G_2$ and $G_1 \times D_i$ are vertex identifying in $G_1 \times G_2$. Indeed, clearly every vertex lies in at least one cycle and if u and v lie in the same set of cycles then their first components must coincide and their second components must coincide too. This proves (i).

The proof of (ii) is similar.

In particular, if both G_1 and G_2 have an identifying collection of cycles, then so has $G_1 \times G_2$. In essentially the same way we can prove

Theorem 6. *Suppose that G_1 and G_2 are Hamiltonian. Furthermore, suppose that $\{P_1,\dots,P_m\}$ (resp. $\{Q_1,\dots,Q_n\}$) is a vertex identifying collection of paths in G_1 (resp. G_2). Then the graphs $P_i \times G_2$ $(i=1,\dots,m)$ and $G_1 \times Q_j$ $(j=1,\dots,n)$ are Hamiltonian subgraphs of $G_1 \times G_2$ and $v(G_1 \times G_2) \leqslant m+n$.*

In the previous theorem the paths $P_1,\dots,P_m$ do not have to identify the empty set; the sets $G_1 \times Q_j$ $(j=1,\dots,n)$ contain every vertex of $G_1 \times G_2$ anyway. So we allow the possibility that there is (at most one) $v \in G_1$ such that $v \notin P_1,\dots,P_m$.

We present two more modifications of Theorem 5.

Lemma 7. *Suppose that G_1 and G_2 have Hamilton paths and both have an even number of vertices. Let $\{P_1,\dots,P_m\}$ (resp. $\{Q_1,\dots,Q_n\}$) be a vertex identifying collection of paths (each of length at least 1) in G_1 (resp. G_2). Then the graphs $P_i \times G_2$ $(i=1,\dots,m)$ and $G_1 \times Q_j$ $(j=1,\dots,n)$ are Hamiltonian subgraphs of $G_1 \times G_2$ and $v(G_1 \times G_2) \leqslant m+n$.*

Proof. This follows from the proof of Theorem 5. □

Theorem 8. *Suppose that G is Hamiltonian and P_n is a path of length n. Then*

$$v(P_n \times G) \leqslant \left\lceil \frac{n+1}{2} \right\rceil + v(G).$$

Proof. A path of length one can be identified with one subset (now we do not have to identify the empty set). Suppose now that $n \geqslant 2$ and denote the vertices of P_n by $1,\dots,n+1$. If n is even, we can use the sets $\{1,2\},\{2,3,4\},\{4,5,6\},\dots,\{n-2,n-1,n\}$ and $\{n,n+1\}$ to identify P_n. If n is odd we can use the sets $\{1,2\},\{2,3,4\},\dots,\{n-3,n-2,n-1\}$ and $\{n-1,n\}$, respectively (again we do not have to identify the empty set). The claim now follows from the proof of Theorem 5. □

Theorems 6–8 are of course valid for closed walks too, and as a matter a fact, in this case the assumptions can be considerably loosened.

Next, we present a new technique to construct identifying collections of closed walks in certain families of graphs using the previous results. This technique can be described as follows.

Suppose that a graph $G=(V,E)$ is given. Firstly, we "contract" some adjacent vertices of G to one vertex, i.e., we treat some vertices in a connected subgraph as one vertex or a cluster (and this is done usually in several places), to obtain a new graph for which we know a vertex identifying collection. This known collection is used to identify the clusters. To construct a vertex identifying collection of closed walks for G, all we have to do is to separate vertices within the clusters. Of course, this has to be done in such a way that we really obtain closed walks in G.

As an application, we give the following theorem. The cycle of length n is denoted by C_n. Exceptionally, we allow the case $n=2$ and then C_2 is a path of length one.

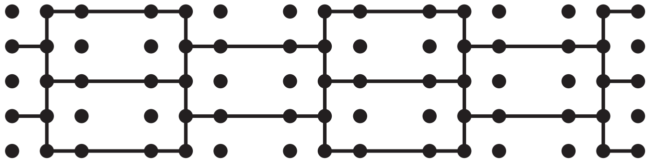

Fig. 1.

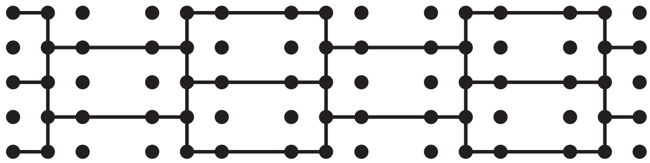

Fig. 2.

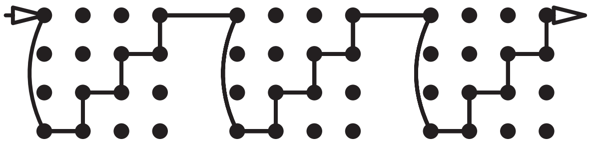

Fig. 3.

Theorem 9. (i) $v^*(C_t \times C_{3t}) \leqslant v^*(C_t \times C_t) + 2$,
(ii) $v^*(C_4 \times C_{4t}) \leqslant v^*(C_4 \times C_t) + 2$,
(iii) $v^*(C_5 \times C_{5t}) \leqslant v^*(C_5 \times C_t) + 3$, *and*
(iv) $v^*(C_6 \times C_{6t}) \leqslant v^*(C_6 \times C_t) + 3$.

Proof. We depict the vertices in $C_m \times C_n$ as an $m \times n$ array. In the case (i) vertices in each row are divided into groups of three. That is, the clusters are paths of length two. A vertex identifying collection for $C_t \times C_t$ is now used to identify the clusters. To separate the vertices in the clusters we can use the two connected subgraphs in Figs. 1 and 2 (we only depict the case $t = 5$).

In case (ii) we proceed similarly. This time the clusters consist of four vertices in a row, i.e., they are paths of length three. To separate the vertices in the clusters we use the cycles in Figs. 3 and 4 (the case $t = 3$ is depicted).

Similarly, in (iii) the clusters consist of five vertices in a row and in (iv) they consist of six vertices in a row. In (iii) the vertices in the clusters are separated using the cycle in Fig. 5 (the case $t = 2$ is depicted) and the two cycles which are obtained from Fig. 5 first shifting it by one step to the right and then shifting it by another step to the right.

To separate the clusters in case (iv) we use the cycle in Fig. 6 (the case $t = 2$ is depicted), then another cycle which is obtained from the cycle in Fig. 6 by shifting

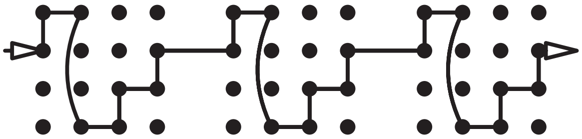

Fig. 4.

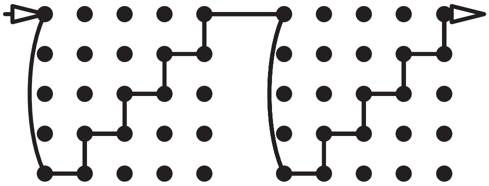

Fig. 5.

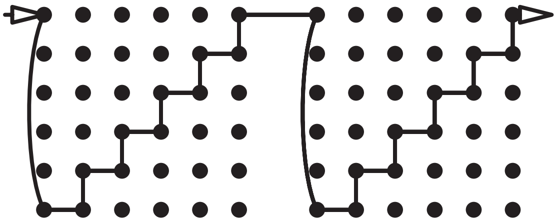

Fig. 6.

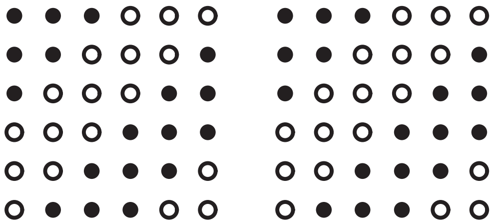

Fig. 7.

it by three steps to the right. The third connected subgraph is specified by the filled circles in Fig. 7.

Of course, many more examples can be given. The results in this section can be used in conjunction with earlier results, e.g., with the fact

$$v(C_n \times C_n) \leqslant 2\lceil \log_2(n+1) \rceil + 1$$

proved in [11].

3. On the identification of edges and turns

For technical reasons we might want to identify edges also. In an interconnection network this corresponds to the situation where a link is faulty. It may also happen that a node is working correctly except between two specific nodes. This leads to the turn (also known as router) identification problem.

The problems of edge and turn identification were posed in [20]. Here we reduce these problems to vertex identification, the central tool being Hamilton decompositions. Another approach to edge identification can be found in [11].

Recall that a turn T in a graph is a path of length two. If $T = uvw$, we call T a *turn at the vertex* v. As an example, if $p, q \geqslant 3$, then there are six turns at each vertex of $C_p \times C_q$.

If $W = v_0 v_1 \cdots v_n$ is a walk in a graph G and $e = uv$ is an edge, we say that W contains e if and only if $u = v_i$ and $v = v_{i+1}$ for some $0 \leqslant i \leqslant n-1$. Similarly, if $T = uvw$ is a turn, we say that W contains T if and only if $u = v_i$, $v = v_{i+1}$ and $w = v_{i+2}$ for some $0 \leqslant i \leqslant n-2$.

The smallest cardinality of an edge identifying collection of closed walks in the binary Hamming space $\mathbb{F}^n$ is denoted by $e^*_H(2, n)$. For a graph G in general this number is denoted by $e^*(G)$. The smallest cardinality of a turn identifying collection of closed walks in a graph G is denoted by $t^*(G)$.

To prove an upper bound for the cardinality of an edge identifying collection of closed walk it suffices to find connected subgraphs such that this bound is valid. In any case, it is a simple task to construct a closed walk from a connected subgraph such that it contains exactly the same edges.

First, we consider identification of edges in the binary Hamming space. Our strategy is to identify a vertex first and then choose an edge incident with it. Using this strategy and Hamilton decompositions we are able to beat the bound $e^*(2, n) \leqslant n + \lfloor \log_2 n \rfloor + 2$ proved in [11]. Another advantage of our approach is that it can be easily generalized.

Recall that a graph G is regular with degree k if every vertex of G is incident with exactly k edges. In this case we write $\deg(G) = k$. Recall also that a *perfect matching* in a graph $G = (V, E)$ is a subset $M \subseteq E$ such that every $v \in V$ is an end of exactly one edge in M. A perfect matching is also known as 1-*factor*.

Definition 10. Let $G = (V, E)$ be a regular graph. It is said to have a *Hamilton decomposition* if either

(i) $\deg(G) = 2d$ and E can be partitioned into d Hamilton cycles, or
(ii) $\deg(G) = 2d + 1$ and E can be partitioned into d Hamilton cycles and a perfect matching.

The following results about Hamilton decompositions can be found in the survey [1]. The results are due to Kotzig [16], Aubert and Schneider [2] and Alspach et al. [1].

Lemma 11. *The binary Hamming space* $\mathbb{F}^n$ *has a Hamilton decomposition for all* n.

Lemma 12. *The product $C_{i_1} \times C_{i_2} \times \cdots \times C_{i_n}$ of cycles has a Hamilton decomposition.*

The following remark is useful.

Remark 13. The vertices of $\mathbb{F}^n$, where $n \geqslant 3$, can be *separated* from each other by using n cycles. That is, there is a set of n cycles which is able to identify any vertex but not the empty set. See Theorem 6 and the discussion following it (see also the proof of Theorem 2 in [11]).

The equality

$$\lceil \log_2(s+1) \rceil = \lfloor \log_2 s \rfloor + 1,$$

which is valid for all positive integers s, is used frequently in the following proof. To see this, choose a k such that $2^k \leqslant s < 2^{k+1}$ and observe that both sides equal $k+1$.

Theorem 14. $e_H^*(2,n) = n + \lfloor \log_2 n \rfloor$, *for all* $n \geqslant 1$.

Proof. It is a simple task to verify the claim for $1 \leqslant n \leqslant 2$, and we now assume that $n \geqslant 3$. Let $\{D_i\}$, where $i = 1, \ldots, n$, be a vertex separating collection of cycles in $\mathbb{F}^n$ (see the remark above). There is one vertex, say x, which is in none of the cycles D_i.

To begin with, we specify a direction to every edge in $\mathbb{F}^n$. To do this, let C_i, where $i = 1, \ldots, \lfloor n/2 \rfloor$, be the cycles in the Hamilton decomposition of $\mathbb{F}^n$ (see Theorem 11). Each C_i is given a direction. If n is odd, then there is a perfect matching in the Hamilton decomposition and the edges in it are directed in such a way that the vertex x has exactly $\lfloor n/2 \rfloor$ outgoing edges. So, if an edge is incident with a vertex v, we can say that it is either an incoming or outgoing edge of v.

From each D_i we construct a subgraph $G_i = (V_i, E_i)$ of $\mathbb{F}^n$ as follows. An edge is in E_i if and only if it is an outgoing edge of any vertex in D_i. The vertex set V_i is defined to consist of those vertices which are incident with an edge in E_i. The subgraphs G_i are obviously connected. They will be used to find the vertex from which the faulty edge is outgoing (if there is one).

To separate the outgoing edges of a vertex we make another use of the Hamilton decomposition. We note that each C_i contains exactly one outgoing edge for each of the vertices. So, if we know the vertex from which the faulty edge is outgoing, we only have to decide in which part of the Hamilton decomposition it is. We consider the cases n even and n odd separately.

Firstly, let n be even. Then the Hamilton decomposition contains $n/2$ cycles. These cycles can be separated from each other using $\lceil \log_2(n/2+1) \rceil$ unions H_i of the cycles in the Hamilton decomposition. Moreover, these unions can be chosen so that they together contain all the edges of $\mathbb{F}^n$, and thus they are able to identify the empty set.

The identification is now done as follows. The unions H_i are used to identify the empty set. Now we may assume that there is an edge, say e, to be identified. The edge e is an outgoing edge of x if and only if it is in none of the subgraphs G_i. Since D_i are vertex separating, we can identify the vertex v such that e is the outgoing edge of it. Now H_i are used to choose the right outgoing edge of v from $n/2$ possible ones.

We have obtained

$$\begin{aligned} e^*_H(2,n) &\leqslant n + \left\lceil \log_2\left(\frac{n}{2}+1\right)\right\rceil \\ &= n + 1 + \left\lfloor \log_2 \frac{n}{2} \right\rfloor \\ &= n + \lfloor \log_2 n \rfloor. \end{aligned}$$

Secondly, let n be odd. We proceed similarly as in the even case. Now the Hamilton decomposition contains $\lfloor n/2 \rfloor$ cycles and a perfect matching. We identify in which *cycle* the edge e is (if in any). This can be done with

$$\begin{aligned} \left\lceil \log_2\left(\left\lfloor \frac{n}{2}\right\rfloor + 1\right)\right\rceil &= \left\lceil \log_2 \frac{n+1}{2} \right\rceil \\ &= \lceil \log_2(n+1)\rceil - 1 \\ &= \lfloor \log_2 n \rfloor \end{aligned}$$

unions H_i of cycles in the Hamilton decomposition. □

The identification is now done as follows. Together G_i and H_i contain every edge of $\mathbb{F}^n$. For this we need the proper orientation of the perfect matching. Thus we can identify the empty set. The edge e is an outgoing edge of x if and only if it is in none of the subgraphs G_i. Thus we can decide the vertex from which e is outgoing. We now use H_i to identify in which cycle of the Hamilton decomposition e is; if e is in none, then it has to be in the perfect matching. But there is at most one outgoing edge per vertex in the perfect matching.

We have shown that also for odd n we have $e^*_H(2,n) \leqslant n + \lfloor \log_2 n \rfloor$.

An easy calculation shows that there are $n2^{n-1}$ edges in $\mathbb{F}^n$ and thus the trivial lower bound of Theorem 2 says that $e^*_H(2,n) \geqslant n + \lfloor \log_2 n \rfloor$. So, we have found the exact value of $e^*_H(2,n)$.

It was proved in [11] that $e^*(C_p \times C_p) \leqslant 2\lceil \log_2 p \rceil + 2$. Our technique allows us to consider arbitrary products of cycles.

Theorem 15. $e^*(C_{q_1} \times \cdots \times C_{q_m}) \leqslant v^*(C_{q_1} \times \cdots \times C_{q_m}) + \lceil \log_2 m \rceil$.

Proof. Our strategy is the same as in the previous theorem. The edges of $C_{q_1} \times \cdots \times C_{q_m}$ are directed using the Hamilton decomposition. From any vertex identifying collection we construct a set of connected subgraphs which contain every edge of $C_{q_1} \times \cdots \times C_{q_m}$ and is able to specify the vertex from which the faulty link (if there is one) goes out. The Hamilton decomposition has m cycles and these can be separated from each other using $\lceil \log_2 m \rceil$ subgraphs (which are unions of some cycles in the decomposition). □

Remark 16. By Theorem 2

$$e^*(C_{q_1} \times \cdots \times C_{q_m}) \geqslant \lfloor \log_2 m q_1 \cdots q_m \rfloor + 1.$$

At the moment, there are no upper bounds for $v^*(C_{q_1} \times \cdots \times C_{q_m})$ except in the case $q_1 = \cdots = q_m$.

We now attack the last problem of this paper: the turn identification. We restrict ourselves to the graphs $C_p \times C_q$, where $p, q \geqslant 3$ and at least one of p and q is even. This guarantees that the cycles in the Hamilton decomposition have an even number of edges.

The following obvious lemma is crucial to the proof of the next theorem.

Lemma 17. *If $G' = (V', E')$ is a connected subgraph of G, then there is a closed walk W in G such that a turn $T = uvw$ is contained in W if and only if $v \in V'$.*

Theorem 18. *Suppose that $p, q \geqslant 3$ and that at least one of p and q is even. Then*

$$t^*(C_p \times C_q) \leqslant v^*(C_p \times C_q) + 3.$$

Proof. Suppose that $G_i = (V_i, E_i)$, where $i = 1, \ldots, t$, is a vertex identifying collection of connected subgraphs of $C_p \times C_q$. From each G_i we construct a closed walk W_i such that a turn $T = uvw$ is contained in W_i if and only if $v \in V_i$.

Every turn of $C_p \times C_q$ is contained in at least one of the closed walks W_i. This follows from the fact that $\{G_i\}$ is vertex identifying. Thus the collection $\{W_i\}$ is able to identify the empty set. Furthermore, if $T = uvw$ is any turn, we are able to identify v.

We will construct closed walks A_i, where $i = 1, 2, 3$, to separate the turns at a vertex. Every A_i contains all the vertices of $C_p \times C_q$. Let C and D be the cycles in the Hamilton decomposition of $C_p \times C_q$. We color every other edge of C with white and every other edge with black. Every other edge of D is coloured red and every other is coloured yellow.

We now specify A_1. First we travel the cycle C, i.e., white and black edges (and thus A_1 contains all white–black turns). After that we continue in such a way that A_1 will contain all white–red and red–black turns. We can travel for example in the order white, red, red, black, red, red, white, red, red, black, red etc.

The closed walk A_2 is constructed similarly to contain all white–black, white–yellow and black–yellow turns. Lastly, A_3 is constructed in such a way that it contains all red–yellow, red–black and yellow–black turns.

It is straightforward to check that A_i are able to separate turns at a vertex.

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